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INDEPENDENCE NUMBERS OF GRAPHS AND GENERATORS OF IDEALS

by

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This article investigates the generators of certain homogeneous ideals which are associated with graphs with bounded independence numbers. These ideals first appeared in the theory of t-designs. The main theorem suggests a new approach to the Clique Problem which is $\mathcal{NP}$-complete. This theorem has a more general form in commutative algebra dealing with ideals associated with unions of linear varieties. This general theorem is stated in the article; a corollary to it generalizes Turán's theorem on the maximum graphs with a prescribed clique number.

1. Introduction

Let G be a graph on n vertices $\{1, ..., n\}$ and let

$$f_G = \prod \{(x_i - x_j)\colon\ i \text{ and } j \text{ are adjacent and } i < j\}$$

be the associated polynomial. Then G has *independence number* $\bar{c}(G)<k$ if and only if at least two of any arbitrarily given k vertices are adjacent. Otherwise said, the polynomial f_G vanishes whenever k variables are set equal. Let, for given integers n and k with $1 \leqq k \leqq n$, $I(k, n)$ denote the ideal of the polynomial ring $\mathbf{Z}[x_1, ..., x_n]$ consisting of the polynomials which vanish whenever k variables are set equal. Then we have

$$(1.1) \qquad \bar{c}(G) < k \quad \text{if and only if} \quad f_G \in I(k, n).$$

The ideal $I(k, n)$ also arises naturally from the study of block t-designs (cf. [1]). In this paper we prove the following theorem concerning generators of $I(k, n)$, as conjectured in [1]:

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Theorem 1. *The ideal $I(k, n)$ is generated by $\Delta(P)$, where $P=\{P_1, \ldots, P_{k-1}\}$ runs through all partitions of the set $\{1, \ldots, n\}$ into $k-1$ (possibly empty) subsets, and*

$$\Delta(P) = \prod_{m=1}^{k-1} \prod_{\substack{i,j \in P_m \\ i<j}} (x_i - x_j).$$

Applying this theorem to f_G, we can restate the criterion (1.1) in

Corollary 1.2. *A graph G has independence number $\tilde{c}(G) \leqq k$ if and only if*

$$f_G = \sum_H g_H \cdot f_H \tag{1.3}$$

where H is the union of k vertex-disjoint complete graphs and g_H is a polynomial.

After examining the degrees of the polynomials f_H, a well-known theorem of Turán [3] can be deduced from this corollary. (See section 4). More importantly, Corollary 1.2 suggests a new way of attacking some of the outstanding problems in graph theory. For instance, one might solve the problem of finding sufficient conditions for $\tilde{c}(G) \leqq k$ which can be verified within polynomial time by looking for graphs G such that the number of H's appearing in (1.3) is bounded by a polynomial in $|G|$. There is also an interesting question of finding an infinite family of graphs G so that the minimal number of H's needed in formula (1.3) is exponential in $|G|$. Such graphs exist if one assumes the hypothesis that $\mathcal{NP} \neq \mathcal{P}$.

The following dual statement is recently proved by D. Kleitman and L. Lovász, using a method similar to our proof of Theorem 1 shown in section 2: A graph G has *chromatic number* $\geqq k$ if and only if f_G lies in the ideal generated by the polynomials f_H where H is a complete k-graph on some subset of vertices of G. It would be quite interesting to study the connection between the representation of f_G in the form (1.3) and its "dual" representation described above.

We remark that the coefficient ring $\mathbf{Z}$ of the polynomials we consider in this paper is irrelevant and can be replaced by any unique factorization domain. A generalization of Theorem 1 in commutative algebra will be proved in a subsequent paper [2]. We only state the result in section 3. Nevertheless, its applications to graph theory concerning the structure of maximal complete k-graph free subgraphs of a given graph, which we call the "Turán property", will be discussed in section 4.

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2. Proof of Theorem 1

First we introduce some notation. Let X be the set $\{x_1, \ldots, x_n\}$ of indeterminates. For a subset Y of X, let $\mathbf{Z}[Y]$ stand for the polynomial ring over $\mathbf{Z}$ with variables in Y. If $f \in \mathbf{Z}[X]$ and $Z \subset X$, denote by f/Z the polynomial obtained from f by setting the variables in Z equal to the first member in Z. Finally, set, for $Y \subset X$,

$$\Delta(Y) = \prod \{(x_i - x_j) \colon x_i, x_j \in Y \text{ and } i < j\}.$$

Obviously, $I(k, n)=0$ when $k=0$ or 1. Therefore we assume $k\geqq 2$ from now on. We shall prove the following

Theorem 2. *For a sequence $X\supset Y_1\supset\dots\supset Y_{k-1}$, define the ideal*

$$I=\{f\in \mathbf{Z}[X]:\ \Delta(Y_1)\dots\Delta(Y_{k-1})\ \text{ divides } f$$
$$\text{and } f/Z=0 \text{ for all } Z\subset X \text{ with } |Z|+\sum_{1<i<k}|Y_i\cap Z|\geqq k\}.$$

Let J be the ideal generated by the polynomials $\Delta(X_1)\dots\Delta(X_{k-1})$, where $X_i\supset Y_i$ and $\bigcup_{1\leqq i<k} X_i=X$. Then $I=J$.

If we take $Y_1=\dots=Y_{k-1}$ to be the empty set, then I is nothing but the ideal $I(k, n)$ and one sees easily that Theorems 1 and 2 coincide in this case.

Proof. We first check that J is contained in I. For this, it suffices to show that each generator $\Delta(X_1)\dots\Delta(X_{k-1})$ is. The condition that $X_i\supset Y_i$ certainly implies that $\Delta(X_1)\dots\Delta(X_{k-1})$ is divisible by $\Delta(Y_1)\dots\Delta(Y_{k-1})$. Moreover, if $Z\subset X$ satisfies

$$|Z|+\sum_{1<i<k}|Y_i\cap Z|\geqq k,$$

then

$$\sum_{1\leqq i<k}|Z\cap X_i|\geqq |Z-Y_1|+\sum_{1\leqq i<k}|Z\cap Y_i|\geqq k$$

since $X-Y_1$ is covered by the X_i's. Therefore $|Z\cap X_i|\geqq 2$ for some i and $\Delta(X_1)\dots$ $\dots\Delta(X_{k-1})/Z=0$.

Now we show $I=J$ by induction on $|X-Y_1|$. The assertion is trivial if $Y_1=X$. Therefore we assume $Y_1\neq X$. We may further assume that $Y_1=\{x_1, \dots, x_r\}$ and each Y_i is a beginning section of this sequence. Let $y\in X-Y_1$. Given $f\in I$, we want to prove that $f\in J$. Our strategy is to find a sequence $f_0=f, f_1, \dots$ of polynomials in $\mathbf{Z}[X]$ such that

(a) $f-f_i$ belongs to J,

(b) $y-x_j$ divides f_i for $1\leqq j\leqq i$.

If we succeed in finding f_r, then we may replace Y_1 by $Y_1\cup\{y\}$, f by f_r and proceed with the induction.

So suppose that f_{i-1} for some $1\leqq i<r$ is defined. Let $g=f_{i-1}/\{x_i, y\}$, then g is a polynomial in $\mathbf{Z}[X-\{y\}]$. Let t be the largest index such that $x_i\in Y_t$ and $x_i\notin Y_{t+1}$. Set $t=k-1$ if $x_i\in Y_{k-1}$. In the latter case, the set $Z=\{x_i, y\}$ satisfies the condition

$$|Z|+\sum_{1<i<k}|Z\cap Y_i|=k,$$

and hence $f_{i-1}/Z=0$, i.e., $y-x_i$ divides f_{i-1}. We simply let $f_i=f_{i-1}$ in this case. Thus we assume $t<k-1$. Suppose $Y_{t+1}=\{x_1, \dots, x_j\}$. Note that $j<i$.

Since

$$(y-x_1)\dots(y-x_{i-1})\Delta(Y_1)\dots\Delta(Y_{k-1})=$$
$$(y-x_{j+1})\dots(y-x_{i-1})\Delta(Y_1)\dots\Delta(Y_{t+1}\cup\{y\})\dots\Delta(Y_{k-1})$$

divides f_{i-1}, the polynomial $\frac{g}{h}$ is divisible by

$$\Delta(Y_1)\ldots\Delta(Y_{t+1}\cup\{x_i\})\ldots\Delta(Y_{k-1}),$$

where $h=(x_i-x_{j+1})\ldots(x_i-x_{i-1})$. Moreover, if any set $Z\subset X-\{y\}$ is such that

$$|Z|+\sum_{\substack{1<i<k\\ i\neq t+1}}|Z\cap Y_i|+|Z\cap(Y_{t+1}\cup\{x_i\})|\geqq k,$$

define the set Z_1 to be either $Z\cup\{y\}$ or Z depending on whether or not $x_i\in Z$; then Z_1 satisfies

$$|Z_1|+\sum_{1<i<k}|Z_1\cap Y_i|\geqq k,$$

and consequently, $f_{i-1}/Z_1=g/Z=0$. Since h^2 divides g, we also have $\frac{g}{h}\Big/Z=0$. Thus we may apply induction (on $|X|$) to $\frac{g}{h}$ and conclude that g is generated by the polynomials $h\Delta(X_1)\ldots\Delta(X_{k-1})$, where $X_i\supset Y_i$, $x_i\in X_{t+1}$ and $\bigcup_{1\leqq i<k}X_i=X-\{y\}$. Write

$$g=\sum u_{X_1\ldots X_{k-1}}h\Delta(X_1)\ldots\Delta(X_{k-1})$$

with $u_{X_1\ldots X_{k-1}}\in\mathbf{Z}[X-\{y\}]$.

Consider the polynomial

$$g_i=\sum u_{X_1\ldots X_{k-1}}(y-x_{j+1})\ldots(y-x_{i-1})\Delta(X_1)\ldots\Delta(\{y\}\cup X_{t+1}-\{x_i\})\ldots\Delta(X_{k-1}).$$

It is clear that $g_i\in J$ and $g_i/\{x_i,y\}=g$. Thus $y-x_i$ divides $f_{i-1}-g_i$. Further, $(y-x_1)\ldots\ldots(y-x_{i-1})$ also divides $f_{i-1}-g_i$ by the assumption (b) on f_{i-1} and the construction of g_i. Putting $f_i=f_{i-1}-g_i$, we are done. ∎

3. Consequences and Generalizations

There are relations among the generators $\Delta(P)$ of Theorem 1. Actually, all the $\Delta(P)$'s can be generated by those ones which have the lowest degree, as shown in the following

Proposition 3.1. *Let* $A=\{x_1,\ldots,x_m\}$ *and* $B=\{x_{m+1},\ldots,x_{2m+1}\}$. *Then*

$$\Delta(A)\cdot\Delta(B)=\sum_{x_b\in B}(-1)^{b+1}\Delta(A\cup\{x_b\})\cdot\Delta(B-\{x_b\}).$$

Consequently, if $C=\{x_{2m+2},\ldots,x_r\}$, *then* $\Delta(A)\cdot\Delta(B\cup C)=$

$$\sum_{x_b\in B}\Big[(-1)^{b+1}\Delta(A\cup\{x_b\})\cdot\Delta(C\cup B-\{x_b\})\cdot\sum_{x_c\in C}(x_b-x_c)\Big].$$

Proof. Put

$$F=\sum_{x_b\in B}(-1)^{b+1}\Delta(A\cup\{x_b\})\Delta(B-\{x_b\}).$$

It is obvious that $\Delta(A)$ divides F since it divides $\Delta(A\cup\{x_b\})$. Given $m+1\leqq i<j\leqq 2m+1$, we claim that x_i-x_j divides F. Indeed, if $b\neq i$ or j, then x_i-x_j divides $\Delta(B-\{x_b\})$; and the sum of the remaining two terms in F

$$(-1)^{i+1}\Delta(A\cup\{x_i\})\Delta(B-\{x_i\})+(-1)^{j+1}\Delta(A\cup\{x_j\})\Delta(B-\{x_j\})$$

is equal to zero (by the definition of Δ) when we set $x_i=x_j$. Thus F is a polynomial divisible by $\Delta(A)\Delta(B)$. Since the degree of F is at most equal to the degree of $\Delta(A)\Delta(B)$, by comparing the coefficients of both polynomials, we see that $F=\Delta(A)\Delta(B)$. ∎

Note that the $\Delta(P)$'s of the lowest degree correspond to the partitions P of the set $\{1, \dots, n\}$ into $k-1$ subsets of as nearly equal sizes as possible. So combining Theorem 1 and Proposition 3.1 together, we have

Corollary 3.2. *The ideal $I(k, n)$ is generated by the polynomials $\Delta(P)$, where P is a partition of $\{1, \dots, n\}$ into $k-1$ subsets of as nearly equal cardinality as possible.*

The theorem below is a generalization of Theorem 1 mentioned in the Introduction.

Theorem 3. *Let Φ be a homogeneous polynomial in $\mathbf{Z}[x_1, \dots, x_n]$ which factors completely into products of the type x_i-x_j. Assume that Φ satisfies the condition*

(3.3) *if x_j-x_m divides Φ, so does x_i-x_m for every $i<j$.*

Then the ideal $I(k, n)\cap\langle\Phi\rangle$ is generated by $\Delta(P)\vee\Phi$, the least common multiple of $\Delta(P)$ and Φ, where $\Delta(P)$ are as in Theorem 1.

The proof of Theorem 3 is similar to the proof of Theorem 1 in spirit but involves a lot more technicalities. This as well as the geometric meaning of Theorem 3 will be given in [2].

4. Applications to Graph Theory

Before discussing the applications of Theorems 1 and 3 to graph theory, we first give a general philosophy on translating problems in graph theory into problems on ideal generators.

All the graphs on the same set of n vertices form a lattice L_n under inclusion. This lattice is isomorphic to the Boolean algebra of the subsets of an $\binom{n}{2}$-element set. A collection $\mathscr{U}$ of graphs is called an *upper ideal* in L_n if whenever a graph G contains a subgraph belonging to $\mathscr{U}$, then G itself belongs to $\mathscr{U}$. Denote by $I_{\mathscr{U}}$ the ideal in $R[x_1, \dots, x_n]$ generated by the associated polynomials f_G, $G\in\mathscr{U}$. Many graph-theoretic problems are concerned with finding the smallest number of edges among all the graphs belonging to an upper ideal $\mathscr{U}$ and also determining all the graphs with this minimum number of edges. In terms of polynomials, this is equivalent to finding the minimum degree among all non-zero polynomials in $I_{\mathscr{U}}$. For this purpose, it suffices to find a set of homogeneous polynomials generating the ideal $I_{\mathscr{U}}$ such that the minimum degree among them is *computable.*

The applications we shall see below are examples of this philosophy.

Given a graph G, let $c(G)$ and $\tilde{c}(G)$ denote the *clique number* and the *independence number* of G, respectively. The complementary graph $\tilde{G}$ of G is the graph on the same set of vertices such that two vertices are adjacent in $\tilde{G}$ if and only if they are not adjacent in G. Thus $\tilde{c}(G)=c(\tilde{G})$. Let K_n denote the complete graph on n vertices.

Fix an integer k with $1<k\leqq n$. Write $n=q(k-1)+r$, where $0\leqq r<k-1$. We know from Corollary 3.2 that every nonzero polynomial in the ideal $I(k, n)$ must have degree $\geqq(k-1)\binom{q}{2}+rq$, which is the common degree of the generators $\Delta(P)$ of $I(k, n)$ described in that corollary. Considering the complementary graphs of the graphs associated with these $\Delta(P)$'s and using the criterion (1.1), we see that a graph with clique number less than k can have at most

$$\binom{n}{2}-(k-1)\binom{q}{2}-rq=\frac{k-2}{2(k-1)}(n^2-r^2)+\binom{r}{2}$$

edges. This is a new proof of a well-known theorem of Turán [3].

Let t be a positive integer. A graph G is said to be *t-partite* (resp. *complete t-partite*) if there is a way of partitioning the vertices into t disjoint subsets $V_1, \dots, V_t$ such that G is contained in (resp. is) the complement of the t complete graphs on the set of vertices in V_i, $1\leqq i\leqq t$. We have

Corollary 4.1. (Turán) *Given an integer k with $1<k\leqq n$, a graph on n vertices with clique number less than k has at most*

$$\frac{k-2}{2(k-1)}(n^2-r^2)+\binom{r}{2}$$

edges, where $0\leqq r<k-1$ and $r\equiv n \pmod{k-1}$. Moreover, the graph achieving this bound is unique (up to isomorphism); it is a complete $(k-1)$-partite graph.

Proof. It remains to prove the second assertion. Let G be a graph with clique number less than k that has the described maximum number of edges. Thus $f_{\tilde{G}}$ is a polynomial in $I(k, n)$ with the minimum possible degree. We want to show that $f_{\tilde{G}}$ is one of the $\Delta(P)$'s of Corollary 3.2. For this, it suffices to prove that there is a partition $P'=\{P_1, \dots\}$ of $\{1, \dots, n\}$ such that $f_{\tilde{G}}=\Delta(P')$, because then there are at most $k-1$ nonempty sets P_i in P' (since $f_{\tilde{G}}\in I(k, n)$) and consequently, P' is the desired partition due to the minimality of degree $\Delta(P')$.

Let i be a vertex and G_i be a maximum complete subgraph of $\tilde{G}$ containing the vertex i. Let V_i be the set of vertices in G_i. Suppose that j is a vertex outside V_i which is adjacent to say i. Then $f_{\tilde{G}}$ is in $I(k, n)\cap\langle\Phi\rangle$, where $\Phi=(x_i-x_j)\Delta(V_i)$. Applying Theorem 3, we see that $f_{\tilde{G}}$ is generated by $\Delta(P)\vee\Phi$. Since the degree of $f_{\tilde{G}}$ is equal to the minimum degree among all $\Delta(P)$, it follows that $f_{\tilde{G}}$ is a linear combination of those $\Delta(P)$ divisible by Φ. Each P being a partition, this means that every $\Delta(P)$, and hence $f_{\tilde{G}}$, is divisible by $\Delta(V_i\cup\{j\})$. This contradicts the maximality of G_i. Therefore G_i is unique. Now letting P' consist of the distinct V_i's, we have $f_{\tilde{G}}=\Delta(P')$, as desired. ∎

Corollary 4.2. *There exists a graph with n vertices and e edges which has the clique number c if and only if*

$$\binom{c}{2} \leqq e \leqq \frac{c-1}{2c}(n^2-r^2)+\binom{r}{2},$$

where $0 \leqq r < c$ *and* $r \equiv n \pmod c$.

Proof. The necessity of the first inequality is obvious. The second inequality follows from Corollary 4.1. On the other hand, given the vertex number n, the edge number e, and the clique number c satisfying these inequalities, the construction of a graph with these parameters is straightforward. ▮

In view of Turán's theorem, we shall say that a graph G has the *Turán property* if, for every integer k, $1 < k \leqq n$, there is, among all K_k-free subgraphs of G, a $(k-1)$-partite subgraph which has maximum number of edges. Thus Corollary 4.1. says that the complete graph K_n has the Turán property. There are graphs, for example, the *pentagon*, which do not have this property. The following theorem gives a wide class of graphs which have the Turán property.

Theorem 4. *Let G be a graph on n vertices, labeled as* $1, \ldots, n$, *satisfying the condition*

(4.3): *if a vertex j is adjacent to a vertex m, so is every vertex i with* $i > j$ *and* $i \neq m$.

Then G has the Turán property.

Note that if a graph satisfies (4.3), then the complementary graph also satisfies the same condition but with reversed labeling on the vertices.

Proof. Write $\Phi = f_{\bar{G}}$. Then the condition (4.3) on G in the theorem is equivalent to the condition (3.3) stated in Theorem 3. Fix an integer $1 < k \leqq n$. A graph H on the same n vertices is a subgraph of G with clique number $c(H) < k$ if and only if

$$f_{\bar{H}} \in I(k, n) \cap \langle \varphi \rangle.$$

Therefore the largest possible number of edges in such a graph H is $\binom{n}{2}$ minus the minimum degree of nonzero polynomials in the ideal $I(k, n) \cap \langle \Phi \rangle$. From Theorem 3, we know that $I(k, n) \cap \langle \Phi \rangle$ is generated by those $f_{\tilde{H}}$, where $\tilde{H}$ is the union of $\tilde{G}$ with $k-1$ disjoint complete subgraphs of K_n. ▮

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